Chapter 1

Linearni diferencni rovnice

Postup:

- (A) Nalezeni obecneho homogenniho reseni $y_h(n)$. Nebo tez prostor reseni homogenni rovnice, ci baze tohoto prostoru.
- (B) Nalezeni partikularniho reseni $y_p(n)$ pomoci metody specialni prave strany.
- (C) Nalezeni obecneho reseni $y(n) = y_h(n) + y_p(n)$. V pripade pocatecni podminky, nalezeni reseni splnujiciho pocatecni podminku.

1.1 Homogenni rovnice

Postup:

- (i) Nalezeni charakteristickeho polynomu $\chi(t)$.
- (ii) Nalzeni korenu charakteristickeho polynomu $\chi(t)$ splolu s nasobnosti techto korenu.
- (iii) Nalezeni baze prostoru reseni homogenni rovnice pomoci vety o tvaru fundamentálního systému řešení homogenní lineární diferenční rovnice k-tého řádu s konstantními koeficienty (V3). Prvky baze maji tvar n-tych mocnin korenu charakteristickeho polynomu. V pripade vycenasobneho korenu se pak jeste prenasobuje n-kem (pripadne vyssi mocninou n-ka).
- (iv) Zapsani $y_h(n)$ jakozto linearni kombinace fundamentalniho systemu z bodu (iii). V pripade pocatecnich podminek, nalezeni reseni vyhovujiciho pocatecnim podminkam.

1.1.1 $y(n+2) + 4y(n+1) + 4y(n) = 0$

- (i) $\chi(t) = t^2 + 4t + 4$,
- (ii) $\{-2, -2\}$,
- (iii) $\{(-2)^n, n(-2)^n\}$,
- (iv) $y_h(n) = a(-2)^n + bn(-2)^n,\ a, b \in \mathbb{R}$.

1.1.2 $y(n+2) - 3y(n+1) + 2y(n) = 0$

- (i) $\chi(t) = t^2 - 3t + 2$,
- (ii) $\{1, 2\}$,
- (iii) $\{1, 2^n\}$,
- (iv) $y_h(n) = a + b2^n$, $a, b \in \mathbb{R}$.

1.1.3 $y(n+2) - 6y(n+1) + 13y(n) = 0$

- (i) $\chi(t) = t^2 - 6t + 13$,
- (ii) $\{3 \pm 2i\}$,
- (iii) $\left\{13^{\frac{n}{2}} \cos\left(\arctan\left(\frac{2}{3}\right) n\right), 13^{\frac{n}{2}} \sin\left(\arctan\left(\frac{2}{3}\right) n\right)\right\}$,
- (iv) $y_h(n) = a13^{\frac{n}{2}} \cos\left(\arctan\left(\frac{2}{3}\right) n\right) + b13^{\frac{n}{2}} \sin\left(\arctan\left(\frac{2}{3}\right) n\right)$, $a, b \in \mathbb{R}$.

1.1.4 $y(n+2) - 2y(n+1) - 3y(n) = 0$, $y(1) = 2$, $y(2) = 1$

- (i) $\chi(t) = t^2 - 2t - 3$,
- (ii) $\{-1, 3\}$,
- (iii) $\{(-1)^n, 3^n\}$,
- (iv) $y_h(n) = a(-1)^n + b3^n$, $a = -\frac{5}{4}, b = \frac{1}{4}$.

1.1.5 $y(n+2) - y(n+1) - y(n) = 0$, $y(1) = y(2) = 1$

- (i) $\chi(t) = t^2 - t - 1$,
- (ii) $\left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\}$,
- (iii) $\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n, \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$,
- (iv) $y_h(n) = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$, $a = \frac{\sqrt{5}}{5}, b = -\frac{\sqrt{5}}{5}$.

1.1.6 $y(n+4) + 6y(n+2) + 9y(n) = 0$

- (i) $\chi(t) = t^4 + 6t^2 + 9$,
- (ii) $\{\pm i\sqrt{3}, \pm i\sqrt{3}\}$,
- (iii) $\left\{3^{\frac{n}{2}} \cos\left(n\frac{\pi}{2}\right), n3^{\frac{n}{2}} \cos\left(n\frac{\pi}{2}\right), 3^{\frac{n}{2}} \sin\left(n\frac{\pi}{2}\right), n3^{\frac{n}{2}} \sin\left(n\frac{\pi}{2}\right)\right\}$,
- (iv) $y_h(n) = 3^{\frac{n}{2}}\left((a+bn)\cos\left(n\frac{\pi}{2}\right) + (c+dn)\sin\left(n\frac{\pi}{2}\right)\right)$, $a, b, c, d \in \mathbb{R}$.

1.1.7 $y(n+6) - 2y(n+3) + 2y(n) = 0$

- (i) $\chi(t) = t^6 - 2t^3 + 2$,
- (ii) $\left\{\sqrt[6]{2}(\cos(\alpha) + i\sin(\alpha));\ \alpha \in \left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}\right\}$,
- (iii) $\left\{2^{\frac{n}{6}} \cos(\alpha), 2^{\frac{n}{6}} \sin(\alpha);\ \alpha \in \left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}\right\}\right\}$,
- (iv) $y_h(n) = 2^{\frac{n}{6}} \sum_{i=1}^{3}(a_i \cos(\alpha_i) + b_i \sin(\alpha_i))$, $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, $\alpha_1 = \frac{\pi}{12}$, $\alpha_2 = \frac{7\pi}{12}$, $\alpha_3 = \frac{3\pi}{4}$.

1.2 Rovnice se specialni pravou stranou

Nejprve resime prislusnou homogenni rovnici (viz predchozi kapitola). Pak nalezneme partikularni reseni. Nakonec nalezneme obecne reseni viz bod (C).

Postup nalezeni partikularniho reseni:

- (a) Nalezeni m, α, ν a $k = \max\{stP, stQ\}$, z vety o specialni prave strane (V5), kde prava strana je rovna $\alpha^n(P(n)\cos(n\nu) + Q(n)\sin(n\nu))$.
- (b) Pomoci k vyjadrime obecne tvary polynomu R, S (napr.: $k = 2$ implikuje $R(n) = an^2 + bn + c$, kde $a, b, c \in \mathbb{R}$. Dosadime tyto obecne polynomy a drive nalezene m, α, ν do vety (V5) a obdrzime obecny tvar partikularniho reseni $y_p(n) = n^m\alpha^n(R(n)\cos(n\nu) + S(n)\sin(n\nu))$.
- (c) Dosadime partikularni reseni do rovnice a dopocitame presny tvar polynomu R a S a tim tez presny tvar $y_p(n)$. Take lze dosadit za n urcita cisla (napr. $0, 1, 2$ atd.) a obdrzet soustavu rovnic. Je potreba dosadit tolik cisel, aby bylo mozno soustavu rovnic vyresit.

1.2.1 $y(n+4) - y(n) = \sin\left(\frac{n\pi}{4}\right)$

- (i) $\chi(t) = t^4 - 1$,
- (ii) $\{\pm 1, \pm i\}$,
- (iii) $\left\{1, (-1)^n, \sin\left(\frac{n\pi}{2}\right), \cos\left(\frac{n\pi}{2}\right)\right\}$,
- (iv) $y_h(n) = A + B(-1)^n + C\sin\left(\frac{n\pi}{2}\right) + D\cos\left(\frac{n\pi}{2}\right)$, $A, B, C, D \in \mathbb{R}$.
- (a) $m = 0, \alpha = 1, \nu = \frac{\pi}{4}, k = 0$.
- (b) $R(n) = a$, $S(n) = b$, $y_p(n) = a\cos\left(\frac{n\pi}{4}\right) + b\sin\left(\frac{n\pi}{4}\right)$, kde $a, b \in \mathbb{R}$.
- (c) $a = 0$, $b = -\frac{1}{2}$, $y_p(n) = -\frac{1}{2}\sin\left(\frac{n\pi}{4}\right)$
- (C) $y(n) = -\frac{1}{2}\sin\left(\frac{n\pi}{4}\right) + A + B(-1)^n + C\sin\left(\frac{n\pi}{2}\right) + D\cos\left(\frac{n\pi}{2}\right)$, $A, B, C, D \in \mathbb{R}$.

1.2.2 $y(n+4) + y(n) = \sin\left(\frac{n\pi}{4}\right)$

- (i) $\chi(t) = t^4 + 1$,
- (ii) $\left\{\frac{1}{\sqrt{2}}(\pm 1 \pm i)\right\}$,
- (iii) $\left\{\sin\left(\frac{n\pi}{4}\right), \cos\left(\frac{n\pi}{4}\right), \sin\left(\frac{3n\pi}{4}\right), \cos\left(\frac{3n\pi}{4}\right)\right\}$,
- (iv) $y_h(n) = A\sin\left(\frac{n\pi}{4}\right) + B\cos\left(\frac{n\pi}{4}\right) + C\sin\left(\frac{3n\pi}{4}\right) + D\cos\left(\frac{3n\pi}{4}\right)$, $A, B, C, D \in \mathbb{R}$.
- (a) $m = 1, \alpha = 1, \nu = \frac{\pi}{4}, k = 0$.
- (b) $R(n) = a$, $S(n) = b$, $y_p(n) = n(a\cos\left(\frac{n\pi}{4}\right) + b\sin\left(\frac{n\pi}{4}\right))$, kde $a, b \in \mathbb{R}$.
- (c) $a = 0$, $b = -\frac{1}{4}$, $y_p(n) = -\frac{1}{4}n\sin\left(\frac{n\pi}{4}\right)$
- (C) $y(n) = -\frac{1}{4}n\sin\left(\frac{n\pi}{4}\right) + A\sin\left(\frac{n\pi}{4}\right) + B\cos\left(\frac{n\pi}{4}\right) + C\sin\left(\frac{3n\pi}{4}\right) + D\cos\left(\frac{3n\pi}{4}\right)$, $A, B, C, D \in \mathbb{R}$.

1.2.3 $y(n+2) - y(n+1) + y(n) = \sin\left(\frac{n\pi}{3}\right)$

- (i) $\chi(t) = t^2 - t + 1$,
- (ii) $\left\{\frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$,
- (iii) $\left\{\cos\left(\frac{n\pi}{3}\right), \sin\left(\frac{n\pi}{3}\right)\right\}$,
- (iv) $y_h(n) = A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right),\ A, B \in \mathbb{R}$.
- (a) $m = 1, \alpha = 1, \nu = \frac{\pi}{3}, k = 0$.
- (b) $R(n) = a,\ S(n) = b,\ y_p(n) = n\left(a\cos\left(\frac{n\pi}{3}\right) + b\sin\left(\frac{n\pi}{3}\right)\right)$, kde $a, b \in \mathbb{R}$.
- (c) $a = -\frac{1}{2\sqrt{3}},\ b = -\frac{1}{2},\ y_p(n) = -n\left(\frac{1}{2\sqrt{3}}\cos\left(\frac{n\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{3}\right)\right)$
- (C) $y(n) = -n\left(\frac{1}{2\sqrt{3}}\cos\left(\frac{n\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{n\pi}{3}\right)\right) + A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right),\ A, B \in \mathbb{R}$.

1.2.4 $y(n+2) - 2y(n+1) + 2y(n) = \cos(n)$

- (i) $\chi(t) = t^2 - 2t + 2$,
- (ii) $\{1+i, 1-i\}$,
- (iii) $\left\{2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right), 2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right)\right\}$,
- (iv) $y_h(n) = A2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right) + B2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right),\ A, B \in \mathbb{R}$.
- (a) $m = 0, \alpha = 1, \nu = 1, k = 0$.
- (b) $R(n) = a,\ S(n) = b,\ y_p(n) = a\cos(n) + b\sin(n)$, kde $a, b \in \mathbb{R}$.
- (c) $a = \frac{\cos(2) - 2\cos(1) + 2}{9 - 12\cos(1) + 4\cos(2)},\ b = \frac{\sin(2) - 2\sin(1)}{9 - 12\cos(1) + 4\cos(2)}$.
- (C) $y(n) = y_p(n) + A2^{\frac{n}{2}}\sin\left(\frac{n\pi}{4}\right) + B2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right),\ A, B \in \mathbb{R}$.

1.2.5 $y(n+2) - 3y(n+1) + 2y(n) = n^2$, $y(1) = 3$, $y(2) = 2$

- (i) $\chi(t) = t^2 - 3t + 2$,
- (ii) $\{1, 2\}$,
- (iii) $\{1, 2^n\}$,
- (iv) $y_h(n) = A + B2^n,\ A, B \in \mathbb{R}$.
- (a) $m = 1, \alpha = 1, \nu = 0, k = 2$.
- (b) $R(n) = an^2 + bn + c,\ y_p(n) = n\left(an^2 + bn + c\right)$, kde $a, b, c \in \mathbb{R}$.
- (c) $a = -\frac{1}{3},\ b = -\frac{1}{2},\ c = -\frac{13}{6}$.
- (C) $y(n) = B2^n - \frac{1}{3}n^3 - \frac{1}{2}n^2 - \frac{13}{6}n + A,\ A = 1,\ B = \frac{5}{2}$.

1.2.6 $y(n+2) - y(n) = 17$, $y(1) = y(2) = 0$

- (i) $\chi(t) = t^2 - 1$,
- (ii) $\{-1, 1\}$,
- (iii) $\{(-1)^n, 1\}$,
- (iv) $y_h(n) = A(-1)^n + B$, $A, B \in \mathbb{R}$.
- (a) $m = 1, \alpha = 1, \nu = 0, k = 0$.
- (b) $R(n) = a$, $y_p(n) = an$, kde $a \in \mathbb{R}$.
- (c) $a = \frac{17}{2}$.
- (C) $y(n) = A(-1)^n + \frac{17}{2}n + B$, $A = -\frac{17}{4}$, $B = -\frac{51}{4}$.

1.3 Rovnice s pravou stranou ve tvaru souctu specialnich pravych stran

V nekterych pripadech nema prava strana specialni tvar, ale ma tvar souctu vice specialnich pravych stran. Tedy, $PS = \sum_{i=1}^{s} f_i(n)$, kde $f_i(n)$ ma specialni tvar (viz veta V5) pro $i = 1, \ldots, s$. V takovemto pripade krome homogenniho reseni $y_h(n)$ spocitame s partikularnich reseni $y_p^i(n)$, $i = 1, \ldots, s$, ktera budou odpovidat prislusnym pravym stranam. Obecne reseni pak bude mit tvar $y(n) = y_h(n) + \sum_{i=1}^{s} y_p^i(n)$.

1.3.1 $y(n+3) - y(n+2) - 2y(n+1) + 2y(n) = n + 2^n$

- (i) $\chi(t) = t^3 - t^2 - 2t + 2$,
- (ii) $\{1, \pm\sqrt{2}\}$,
- (iii) $\left\{1, (-\sqrt{2})^n, 2^{\frac{n}{2}}\right\}$,
- (iv) $y_h(n) = A + B(-\sqrt{2})^n + C2^{\frac{n}{2}}$, $A, B, C \in \mathbb{R}$.

$f_1(n) = n:$

- (a_1) $m = 1, \alpha = 1, \nu = 0, k = 1$.
- (b_1) $R(n) = an + b$, $y_p(n) = n(an + b)$, kde $a, b \in \mathbb{R}$.
- (c_1) $a = -\frac{1}{2}$, $b = -\frac{3}{2}$.

$f_2(n) = 2^n:$

- (a_2) $m = 0, \alpha = 2, \nu = 0, k = 0$.
- (b_2) $R(n) = a$, $y_p(n) = a2^n$, kde $a \in \mathbb{R}$.
- (c_2) $a = \frac{1}{2}$.
- (C) $y(n) = y_h(n) + y_p^1(n) + y_p^2(n) = A + B(-\sqrt{2})^n + C2^{\frac{n}{2}} - \frac{1}{2}n^2 - \frac{3}{2}n + 2^{n-1}$.

1.3.2 $8y(n+3) + y(n) = 3n + 2^{-n}$

- (i) $\chi(t) = 8t^3 + 1$,
- (ii) $\left\{-\frac{1}{2}, \frac{1}{2}\left(\cos\left(\pm\frac{\pi}{3}\right) + i\sin\left(\pm\frac{\pi}{3}\right)\right)\right\}$,
- (iii) $\left\{\left(-\frac{1}{2}\right)^n, 2^{-n}\cos\left(\pm\frac{n\pi}{3}\right), 2^{-n}\sin\left(\pm\frac{n\pi}{3}\right)\right\}$,
- (iv) $y_h(n) = A\left(-\frac{1}{2}\right)^n + B2^{-n}\cos\left(\pm\frac{n\pi}{3}\right) + C2^{-n}\sin\left(\pm\frac{n\pi}{3}\right),\ A, B, C \in \mathbb{R}$.

$f_1(n) = 3n:$

- (a_1) $m = 0, \alpha = 1, \nu = 0, k = 1$.
- (b_1) $R(n) = an + b$, $y_p(n) = an + b$, kde $a, b \in \mathbb{R}$.
- (c_1) $a = \frac{1}{3}$, $b = -\frac{8}{9}$.

$f_2(n) = 2^{-n}:$

- (a_2) $m = 0, \alpha = \frac{1}{2}, \nu = 0, k = 0$.
- (b_2) $R(n) = a$, $y_p(n) = a2^{-n}$, kde $a \in \mathbb{R}$.
- (c_2) $a = \frac{1}{2}$.
- (C) $y(n) = y_h(n) + y_p^1(n) + y_p^2(n) = A\left(-\frac{1}{2}\right)^n + B2^{-n}\cos\left(\pm\frac{n\pi}{3}\right) + C2^{-n}\sin\left(\pm\frac{n\pi}{3}\right) + \frac{1}{3}n - \frac{8}{9} + 2^{-n-1}$.